Indag. Mathem., N.S., 7 (1), 37–46 March 25, 1996

Perturbing analytic discs attached to maximal real submanifolds of $\mathbb{C}^N$

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Communicated by Prof. J. Korevaar at the meeting of March 27, 1995

ABSTRACT

Let f be an analytic disc in $\mathbb{C}^N$ attached to a maximal real submanifold M of $\mathbb{C}^N$. In a recent paper the author introduced partial indices κ_j, $1 \le j \le N$, of M along the boundary of f and showed that if $\kappa_j \ge 0$ for all j then the family of nearby analytic discs attached to M depends on $\kappa_1 + \cdots + \kappa_N$ parameters. Y.-G. Oh sharpened this by proving the same when $\kappa_j \ge -1$ for all j and showed that in terms of stability this is the best possible condition. In the present paper we explain why the latter condition is natural and give a simple proof of Oh's result in the orientable case.

1. INTRODUCTION

Denote by Δ the open unit disc in $\mathbb{C}$. A submanifold M of $\mathbb{C}^N$ is called *totally real* if for no $x \in M$ the tangent space of M at x contains a complex line. If, in addition, $\dim M = N$ then M is called *maximal real*. A continuous map $f : \bar{\Delta} \to \mathbb{C}^N$, holomorphic on Δ is called an *analytic disc*. We say that *f is attached to M* if its boundary is contained in M, that is, if $f(b\Delta) \subset M$.

Let M be a maximal real submanifold of $\mathbb{C}^N$ and let f be an analytic disc attached to M. In [Gl] we assumed that in a neighbourhood $V \subset \mathbb{C}^N$ of $f(b\Delta)$ we have

$$(1.1) \qquad M \cap V = \{x \in V : \rho(x) = 0\}$$

where $\rho = (\rho_1, \ldots, \rho_N)$ is of class C^2 on V, $\partial\rho_1 \wedge \cdots \wedge \partial\rho_N \ne 0$ on V, and then studied the existence of nearby analytic discs attached to M. We showed that the existence of nearby analytic discs is related to *partial indices of M along* $f \mid b\Delta$ which we introduced as follows. For each $\zeta \in b\Delta$ let $T(\zeta)$ be the tangent space of M at $f(\zeta)$. As f is of class C^{2-0} on $\bar{\Delta}$ [Ch] there is a smooth map $A : b\Delta \to GL(N, \mathbb{C})$ such that for each $\zeta \in b\Delta$ the columns of $A(\zeta)$ form a basis of $T(\zeta)$. Let $B(\zeta) = A(\zeta)\overline{A(\zeta)}^{-1}$ $(\zeta \in b\Delta)$ where bar denotes complex conjugation. Since the spaces $T(\zeta)$ are totally real the map B depends only on the bundle $\{T(\zeta) : \zeta \in b\Delta\}$. By a theorem that follows from the fundamental work of Plemelj [Pl, Ve1] and is sometimes called the Birkhoff factorization theorem [PS, Bi] one can write

$$(1.2) \qquad B(\zeta) = F_1(\zeta)\Lambda(\zeta)F_2(\zeta) \quad (\zeta \in b\Delta)$$

where $F_1 : \bar{\Delta} \to GL(N, \mathbb{C})$ is smooth and holomorphic on Δ, $F_2 : [\mathbb{C} \cup \{\infty\}] \setminus \Delta \to GL(N, \mathbb{C})$ is smooth and holomorphic on $[\mathbb{C} \cup \{\infty\}] \setminus \bar{\Delta}$ and

$$(1.3) \qquad \Lambda(\zeta) = \begin{pmatrix} \zeta^{\kappa_1} & 0 & \cdots & \cdots & 0 \\ 0 & \zeta^{\kappa_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \zeta^{\kappa_N} \end{pmatrix}$$

where κ_j are integers which, modulo a permutation, are the same for all factorizations of B of the form (1.2). We call them *the partial indices of the (trivial) bundle* $\{T(\zeta) : \zeta \in b\Delta\}$ or *the partial indices of M along* $f \,|\, b\Delta$, and we call their sum κ *the total index of T* or *the total index of M along* $f \,|\, b\Delta$. The triviality of the bundle $\{T(\zeta) : \zeta \in b\Delta\}$, that is, the existence of a continuous global basis map $A : b\Delta \to GL(N, \mathbb{C})$ implies that the total index κ which, by (1.2) is the winding number of the map $\zeta \to \det(B(\zeta))$ around 0, equals twice the winding number of $\zeta \to \det(A(\zeta))$ around 0. In particular, κ is even.

A result of Vekua [Ve1] implies that when a map B has the specific form $B = A\overline{A}^{-1}$ as above then there is a smooth map $\Theta : \bar{\Delta} \to GL(N, \mathbb{C})$, holomorphic on Δ and such that

$$(1.4) \qquad B(\zeta) = \Theta(\zeta)\Lambda(\zeta)\overline{\Theta(\zeta)}^{-1} \quad (\zeta \in b\Delta).$$

This implies the existence of a normal form of the bundle $\{T(\zeta) : \zeta \in b\Delta\}$ which we used in [Gl] to generalize a result of Forstnerič [Fo] from $\mathbb{C}^2$ to $\mathbb{C}^N$. We proved that if all partial indices of M along $f \,|\, b\Delta$ are nonnegative then the family of nearby analytic discs depends on $\kappa + N$ real parameters and this still holds for small perturbations of (ρ, the defining function of) M.

Motivated by problems from symplectic geometry, Y.-G. Oh wrote the paper [O1] in which he strenghtened these results. He observed that the above definition of partial indices makes sense also in the case when the bundle $\{T(\zeta) : \zeta \in b\Delta\}$ is not trivial since for each $\zeta \in b\Delta$ the matrix $B(\zeta) = A(\zeta)\overline{A(\zeta)}^{-1}$ depends only on $T(\zeta)$ and not on $A(\zeta)$ whose column vectors form a basis of $T(\zeta)$. Among other things he then proved that for normalized analytic discs the result above about nearby discs holds without the assumption that M be orientable along $f(b\Delta)$, and already as soon as the partial indices satisfy $\kappa_j \geq -1$ $(1 \leq j \leq N)$ rather than $\kappa_j \geq 0$ $(1 \leq j \leq N)$. He proved that for each M' sufficiently close to M the normalized analytic discs sufficiently close to (normalized) f which are attached to M' form a smooth manifold of dimension $\kappa + N$.

(An analytic disc is called *normalized* if it cannot be written in the form $f = g \circ B$ where g is an analytic disc and B an Blaschke product with multiplicity greater than one.)

2. NOTATION AND SOME GENERAL FACTS

Given an open set $V \subset \mathbb{C}^N$ we denote by $\mathcal{C}^2(V)$ the Banach space of all real valued functions of class $\mathcal{C}^2$ with the standard norm

$$\|r\| = \sum_{|\nu| \leq 2} \sup\{|D^\nu r(w)| : w \in V\}$$

where the derivatives are taken with respect to the real coordinates in $\mathbb{C}^N$. Let $0 < \alpha < 1$. We denote by $\mathcal{C}^\alpha$ the Banach algebra of all real valued functions on $b\Delta$ with finite Lipschitz norm of exponent α

$$\|\varphi\|_\alpha = \sup|\varphi(e^{is})| + \sup \frac{|\varphi(e^{is}) - \varphi(e^{it})|}{|e^{is} - e^{it}|^\alpha}, \tag{2.1}$$

by $\mathcal{C}^\alpha_{\mathbb{C}}$ the algebra of complex valued functions on $b\Delta$ with finite norm (2.1) and by $\mathcal{A}^\alpha$ the closed subalgebra of all $\varphi \in \mathcal{C}^\alpha_{\mathbb{C}}$ which extend holomorphically to Δ. The Banach space $(\mathcal{A}^\alpha)^N$ will be the space of analytic discs we will work with. Given $\varphi \in \mathcal{C}^\alpha$ we will denote by $\tilde{\varphi}$ the harmonic conjugate function that vanishes at 0. It is known that $\tilde{\varphi} \in \mathcal{C}^\alpha$ and that $\varphi \mapsto \tilde{\varphi}$ is a bounded linear map of $\mathcal{C}^\alpha$ into itself.

Let X, Y, Z be Banach spaces, let $U \subset X \times Y$ be a neighbourhood of 0 and let $F : U \to Z$ be a map of class $\mathcal{C}^1$, $F(0) = 0$. Denote by $D_Y F$ the partial derivative with respect to Y. Assume that the map $(D_Y F)(0) : Y \to Z$ is *onto* and that $\mathrm{Ker}(D_Y F)(0)$ is complemented in Y, that is, $Y = \mathrm{Ker}(D_Y)(F)(0) \oplus W$ where W is a closed subspace of Y. Identify $X \times Y$ with $[X \times \mathrm{Ker}(D_Y F)(0)] \times W$ in the obvious way. Since $(D_W F)(0) = (D_Y F)(0) \,|\, W$ it follows that $(D_W F)(0) : W \to Y$ is a Banach space isomorphism so by the standard implicit mapping theorem [Ca] there are neighbourhoods $\mathcal{P}_1$ of 0 in X, $\mathcal{P}_2$ of 0 in $\mathrm{Ker}(D_Y F)(0)$, $\mathcal{P}_3$ of 0 in W and a $\mathcal{C}^1$ map $\Phi : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathcal{P}_3$ such that if $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$ then

$$x = (x_1, x_2, x_3) \in \mathcal{P} \quad \text{and} \quad F(x) = 0$$

is equivalent to

$$(x_1, x_2) \in \mathcal{P}_1 \times \mathcal{P}_2 \quad \text{and} \quad x_3 = \Phi(x_1, x_2).$$

In particular, $\{x \in \mathcal{P} : F(x) = 0\}$ is a $\mathcal{C}^1$ submanifold of $\mathcal{P}$ and for each $x_1 \in \mathcal{P}_1$, $\{(x_2, x_3) \in \mathcal{P}_2 \times \mathcal{P}_3 : F(x_1, x_2, x_3) = 0\}$ is a $\mathcal{C}^1$ submanifold of $\mathcal{P}_2 \times \mathcal{P}_3$.

3. MOTIVATION AND THE MAIN RESULT

The simple method used in [Gl] suggested the conditions $\kappa_j \geq 0$ $(1 \leq j \leq N)$ for partial indices of M along $f \,|\, b\Delta$ as natural ones for studying nearby analytic discs. A more sophisticated approach of Oh [Ol] produced the conditions

$$(3.1) \qquad \kappa_j \geq -1 \quad (1 \leq j \leq N)$$

which, at least from the stability point of view, are the correct ones since Oh proved that if one of the partial indices satisfies $\kappa_j \leq -2$ then there are arbitrarily small perturbations M' of M such that near f there are no analytic discs attached to M'. His elegant proofs cannot be understood without studying his paper [O2] devoted to a slightly different topic since they use results from there. Quite recently M. Černe [Če] showed how to treat the case (3.1) by the method from [Gl].

The author tried to get a simple explanation why the conditions (3.1) are natural. The result of this attempt is presented below. Beside answering the question, it provides, at least in the orientable case, a simple proof of the existence result for nearby analytic discs as in [O1] but for general rather than for normalized discs. The proof uses only the factorization (1.4) and the implicit mapping theorem in appropriately chosen function spaces.

Let M be as in (1.1). Passing to a smaller V we may assume that $\partial\rho_1 \wedge \cdots \wedge \partial\rho_N \neq 0$ on a neighbourhood of the compact set $\bar{V}$. Then there is a neighbourhood $\mathcal{S} \subset [\mathcal{C}^2(V)]^N$ of ρ such that for each $r \in \mathcal{S}$

$$M_r = \{x \in V : r(x) = 0\}$$

is a maximal real submanifold of V. In particular, $M_\rho = M$.

Assume that $f \in (\mathcal{A}^\alpha)^N$ satisfies $f(b\Delta) \subset M$, that is, $\rho(f(\zeta)) = 0$ $(\zeta \in b\Delta)$. If $U \subset (\mathcal{A}^\alpha)^N$ is a neighbourhood of f so small that $g(b\Delta) \subset V$ for each $g \in U$ then the analytic disc $g \in U$ is attached to M if and only if $\rho(g(\zeta)) = 0$ $(\zeta \in b\Delta)$. We now follow Baouendi, Rothschild and Trepreau [BRT] by studying the map Q that takes $g \in U$ to the map $\zeta \mapsto \rho(g(\zeta))$. Since ρ is of class $\mathcal{C}^2$ a result of Hill and Taiani [HT] implies that Q is a $\mathcal{C}^1$ map from U to $(\mathcal{C}^\alpha)^N$, and $\mathcal{M} = \{g \in U : Q(g) = 0\}$ is precisely the set of all analytic discs in U attached to M. By the implicit mapping theorem $\mathcal{M}$ will be a $\mathcal{C}^1$ manifold if $(DQ)(f)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$ and if $\mathrm{Ker}(DQ)(f)$ is complemented in $(\mathcal{A}^\alpha)^N$. We shall see that (1.4) implies that $\mathrm{Ker}(DQ)(f)$ is always finite dimensional and thus complemented in $(\mathcal{A}^\alpha)^N$. Thus we have to require only that $(DQ)(f)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$. It turns out that this is equivalent to (3.1):

Theorem 3.1. *Let M, f and Q be as above. Let $\kappa_1, \ldots, \kappa_N$ be the partial indices of M along $f \,|\, b\Delta$ and let κ be the total index. Then $(DQ)(f)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$ if and only if $\kappa_j \geq -1$ $(1 \leq j \leq N)$. If this happens then there are a neighbourhood $\mathcal{P} \subset [\mathcal{C}^2(V)]^N$ of ρ and a neighbourhood $W \subset (\mathcal{A}^\alpha)^N$ of f such that*

(i) $\{(r,g) \in \mathcal{P} \times W : g(b\Delta) \subset M_r\}$ is a $\mathcal{C}^1$ submanifold of $\mathcal{P} \times W$;

(ii) for each $r \in \mathcal{P}$ the set $\{g \in W : g(b\Delta) \subset M_r\}$ is a $\mathcal{C}^1$ submanifold of W of dimension $\kappa + n$.

Note that (ii) says that for each M' sufficiently close to M the family of analytic discs in W attached to M' is a $(\kappa + N)$-dimensional submanifold of W and (i) says in particular that these manifolds depend smoothly on M.

4. PROOF OF THE FIRST PART OF THEOREM 3.1: PARTIAL INDICES AND THE SURJECTIVITY OF THE DERIVATIVE

Let $\kappa_1, \kappa_2, \ldots, \kappa_N$ be the partial indices of M along $f \,|\, b\Delta$. Write $w \in (\mathcal{A}^\alpha)^N$ as a column. Then

$$[(DQ)(f)(w)](\zeta) = 2\Re[G(\zeta)w(\zeta)] \quad (\zeta \in b\Delta)$$

where

$$G(\zeta) = \begin{pmatrix} \frac{\partial \rho_1}{\partial z_1}(f(\zeta)) & \cdots & \frac{\partial \rho_1}{\partial z_N}(f(\zeta)) \\ \cdots & \cdots & \cdots \\ \frac{\partial \rho_N}{\partial z_1}(f(\zeta)) & \cdots & \frac{\partial \rho_N}{\partial z_N}(f(\zeta)) \end{pmatrix} \quad (\zeta \in b\Delta).$$

Suppose that $V : b\Delta \to GL(N, \mathbb{R})$ is of class $\mathcal{C}^\alpha$. Then the map $\varphi \mapsto V\varphi$ is a Banach space isomorphism of $(\mathcal{C}^\alpha)^N$ onto itself. As V is real we have $V(\zeta)\Re[G(\zeta)w(\zeta)] = \Re[V(\zeta)G(\zeta)w(\zeta)]$ ($\zeta \in b\Delta$) and thus $(DQ)(f)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$ if and only if the map taking $w \in (\mathcal{A}^\alpha)^N$ to $\zeta \mapsto \Re[V(\zeta)G(\zeta)w(\zeta)]$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$. We now choose V so that this is easy to check.

Let $A : b\Delta \to GL(N, \mathbb{C})$ be a map of class $\mathcal{C}^\alpha$ such that for each $\zeta \in b\Delta$ the columns of $A(\zeta)$ form a basis of $T(\zeta)$, the tangent space of M at $f(\zeta)$. Clearly $\Re[G(\zeta)A(\zeta)] = 0$ ($\zeta \in b\Delta$) which implies that

$$A(\zeta)\overline{A(\zeta)}^{-1} = -G(\zeta)^{-1}\overline{G(\zeta)} \quad (\zeta \in b\Delta). \tag{4.1}$$

We want to find V so that $\zeta \mapsto H(\zeta) = V(\zeta)G(\zeta)$ is as simple as possible. Suppose that a map $H : b\Delta \to GL(N, \mathbb{C})$ satisfies $G(\zeta)^{-1}\overline{G(\zeta)} = H(\zeta)^{-1}\overline{H(\zeta)}$ ($\zeta \in b\Delta$). Then $H(\zeta)G(\zeta)^{-1} = \overline{H(\zeta)G(\zeta)^{-1}} = V(\zeta)$ is real for each $\zeta \in b\Delta$ and $H(\zeta) = V(\zeta)G(\zeta)$ ($\zeta \in b\Delta$). Our assumption together with (4.1) implies that we must choose H so that $H(\zeta)^{-1}\overline{H(\zeta)} = -A(\zeta)\overline{A(\zeta)}^{-1}$ ($\zeta \in b\Delta$). By (1.4) there is a map $\Theta : \bar{\Delta} \to GL(N, \mathbb{C})$ of class $\mathcal{C}^\alpha$, holomorphic on Δ, such that $A(\zeta)\overline{A(\zeta)}^{-1} = \Theta(\zeta)\Lambda(\zeta)\overline{\Theta(\zeta)}^{-1}$ ($\zeta \in b\Delta$) where Λ is as in (1.3). Thus, if a map $M : b\Delta \to GL(N, \mathbb{C})$ satisfies

$$\Lambda(\zeta) = M(\zeta)^{-1}\overline{M(\zeta)} \quad (\zeta \in b\Delta) \tag{4.2}$$

then

$$-A(\zeta)\overline{A(\zeta)}^{-1} = [i\Theta(\zeta)M(\zeta)^{-1}] \cdot \overline{[i\Theta(\zeta)M(\zeta)^{-1}]^{-1}} \quad (\zeta \in b\Delta)$$

and thus $H(\zeta) = M(\zeta)[i\Theta(\zeta)]^{-1}$ ($\zeta \in b\Delta$) has the properties we seek.

We find M satisfying (4.2) as in [Gl]: Observe first that since κ is even the number of odd partial indices κ_j is even. With no loss of generality assume that $\kappa_j = 2m_j - 1$ ($1 \le j \le 2\ell$), $\kappa_j = 2m_j$ ($2\ell + 1 \le j \le N$). Here is a decomposition (4.2) in $GL(1, \mathbb{C})$:

$$\zeta^{2k} = (\zeta^{-k})^{-1}\overline{(\zeta^{-k})} \quad (\zeta \in b\Delta)$$

and here is a decomposition (4.2) in $GL(2, \mathbb{C})$ when both particl indices are odd:

$$\begin{pmatrix} \zeta^{2j-1} & 0 \\ 0 & \zeta^{2k-1} \end{pmatrix} = P(\zeta)^{-1}\overline{P(\zeta)} \quad (\zeta \in b\Delta)$$

where

$$P(\zeta)=\begin{pmatrix}1+\zeta & -i(1-\zeta)\\ i(1-\zeta) & 1+\zeta\end{pmatrix}\begin{pmatrix}\zeta^{-j} & 0\\ 0 & \zeta^{-k}\end{pmatrix}\quad (\zeta\in b\Delta).$$

Using this one can obtain M satisfying (4.2) in a simple way: In the last $N-2\ell$ rows and $N-2\ell$ columns all entries $M_{jk}(\zeta)$ of $M(\zeta)$ vanish except

$$M_{kk}(\zeta)=\zeta^{-m_k}\quad (\zeta\in b\Delta).$$

In the first 2ℓ rows and 2ℓ columns all entries vanish except the ones in ℓ 2×2 matrices on the diagonal; in particular, if $1\le k\le \ell$ then

$$\begin{aligned}&\begin{pmatrix}M_{2k-1,2k-1}(\zeta) & M_{2k-1,2k}(\zeta)\\ M_{2k,2k-1}(\zeta) & M_{2k,2k}(\zeta)\end{pmatrix}\\ &\quad=\begin{pmatrix}1+\zeta & -i(1-\zeta)\\ i(1-\zeta) & 1+\zeta\end{pmatrix}\begin{pmatrix}\zeta^{-m_{2k-1}} & 0\\ 0 & \zeta^{-m_{2k}}\end{pmatrix}.\end{aligned}$$

Denote this matrix by $K_k(\zeta)$.

The preceding discussion implies that $(DQ)(f)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$ if and only if the map taking $w\in(\mathcal{A}^\alpha)^N$ to $\zeta\mapsto \Re[M(\zeta)[i\Theta(\zeta)]^{-1}w(\zeta)]$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$. Since $\Theta:\bar\Delta\to GL(N,\mathbb{C})$ is of class $\mathcal{C}^\alpha$ and holomorphic on Δ the map taking $w\in(\mathcal{A}^\alpha)^N$ to $\zeta\mapsto[i\Theta(\zeta)]^{-1}w(\zeta)$ is a Banach space isomorphism of $(\mathcal{A}^\alpha)^N$ onto itself. Thus, $(DQ)(f)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$ if and only if the map taking $w\in(\mathcal{A}^\alpha)^N$ to $\zeta\mapsto\Re[M(\zeta)w(\zeta)]$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$. Because of the special form of the matrix M this happens if and only if

(i) for each k, $1\le k\le\ell$, the map taking $w\in(\mathcal{A}^\alpha)^2$ to $\zeta\mapsto\Re[K_k(\zeta)w(\zeta)]$ maps $(\mathcal{A}^\alpha)^2$ onto $(\mathcal{C}^\alpha)^2$;

(ii) for each k, $2\ell+1\le k\le N$, the map taking $w\in\mathcal{A}^\alpha$ to $\zeta\mapsto\Re[\zeta^{-m_k}w]$ maps $\mathcal{A}^\alpha$ onto $\mathcal{C}^\alpha$.

For each $\varphi\in\mathcal{C}^\alpha$, $w=\varphi+i\tilde\varphi\in\mathcal{A}^\alpha$ satisfies $\varphi(\zeta)=\Re[\zeta^{-k}(\zeta^k w(\zeta))]$ $(\zeta\in b\Delta)$ for every integer k. If $k\ge 0$ then $\zeta\mapsto\zeta^k w(\zeta)$ belongs to $\mathcal{A}^\alpha$ which shows that the map in (ii) is onto if $k\ge0$. If $k<0$ then $\int_0^{2\pi}e^{-ik\theta}w(e^{i\theta})\,d\theta=0$ for every $w\in\mathcal{A}^\alpha$ and so every $\varphi\in\mathcal{C}^\alpha$ of the form $\zeta\mapsto\Re[\zeta^{-k}w(\zeta)]$, $w\in\mathcal{A}^\alpha$, satisfies $\int_0^{2\pi}\varphi(e^{i\theta})\,d\theta=0$; consequently the map (ii) is not onto. Thus, (ii) holds if and only if $m_k\ge 0$ $(2\ell+1\le k\le N)$.

To consider (i) let

$$K(\zeta)=\begin{pmatrix}1+\zeta & -i(1-\zeta)\\ i(1-\zeta) & 1+\zeta\end{pmatrix}\begin{pmatrix}\zeta^{j} & 0\\ 0 & \zeta^{k}\end{pmatrix}$$

and suppose that given $\varphi\in(\mathcal{C}^\alpha)^2$ we want to find $w\in(\mathcal{A}^\alpha)^2$ such that $\Re[K(\zeta)w(\zeta)]=\varphi(\zeta)$ $(\zeta\in b\Delta)$, that is,

$$\Re\left(\begin{pmatrix}1+\zeta^2 & -i(1-\zeta^2)\\ i(1-\zeta^2) & 1+\zeta^2\end{pmatrix}\begin{pmatrix}\zeta^{2j}w_1(\zeta^2)\\ \zeta^{2k}w_2(\zeta^2)\end{pmatrix}\right)=\begin{pmatrix}\varphi_1(\zeta^2)\\ \varphi_2(\zeta^2)\end{pmatrix}\quad(\zeta\in b\Delta),$$

that is

$$2\Re\begin{pmatrix}\zeta\cdot\zeta^{2j}w_1(\zeta^2)\\ \zeta\cdot\zeta^{2k}w_2(\zeta^2)\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\varphi_1(\zeta^2)\\ \varphi_2(\zeta^2)\end{pmatrix}\tag{4.3}$$

where $\zeta=e^{i\theta}$. Call a function φ on $b\Delta$ *odd* if $\varphi(-\zeta)=-\varphi(\zeta)$ $(\zeta\in b\Delta)$.

Observe that $\varphi \in \mathcal{A}^\alpha$ is odd if and only if $\varphi(\zeta) = \zeta\psi(\zeta^2)$ $(\zeta \in \Delta)$ with $\psi \in \mathcal{A}^\alpha$, and that if $\varphi \in \mathcal{C}^\alpha$ is odd then $\tilde{\varphi}$ is also odd. Let $k = j = 0$. Observe that if $\varphi \in \mathcal{C}^\alpha$ is odd then $\varphi + i\tilde{\varphi}$ is the unique odd function in $\mathcal{A}^\alpha$ whose real part is φ. Since in (4.3) the function on the right is odd and the function in the bracket on the left is odd it follows that for every $\varphi \in (\mathcal{C}^\alpha)^2$ there is a unique $w \in (\mathcal{A}^\alpha)^2$ such that (4.3) is satisfied with $k = j = 0$.

Let $k \le 0$, $j \le 0$. Write $\zeta w_1(\zeta^2) = \zeta \cdot \zeta^{2j}[\zeta^{-2j} w_1(\zeta^2)]$, $\zeta w_2(\zeta) = \zeta \cdot \zeta^{2k}[\zeta^{-2k} w_2(\zeta^2)]$ and observe that the functions in brackets belong to $\mathcal{A}^\alpha$. The preceding discussion now implies that the map $w \mapsto (\zeta \mapsto \Re[K(\zeta)w(\zeta)])$ maps $(\mathcal{A}^\alpha)^2$ onto $(\mathcal{C}^\alpha)^2$.

Let $j \ge 1$. Choose $g \in \mathcal{A}^\alpha$, $g(0) \ne 0$. Write $g = \varphi_1 - i\varphi_2$. Then

$$\Re[\zeta g(\zeta^2)] = \varphi_1(\zeta^2)\cos\theta + \varphi_2(\zeta^2)\sin\theta \tag{4.4}$$

where $\zeta = e^{i\theta}$. By the preceding discussion $\zeta \mapsto \zeta g(\zeta^2)$ is the only function in $\mathcal{A}^\alpha$ whose real part is the term on the right in (4.4). Since $g(0) \ne 0$ and since $j \ge 1$, $\zeta g(\zeta^2)$ cannot be written in the form $\zeta \cdot \zeta^{2j} g(\zeta^2)$. It follows that the map $w \mapsto (\zeta \mapsto \Re[K(\zeta)w(\zeta)])$ does not map $(\mathcal{A}^\alpha)^2$ onto $(\mathcal{C}^\alpha)^2$. The same reasoning applies if $k \ge 1$. This proves that (i) holds if and only if $m_k \ge 0$ $(1 \le k \le 2\ell)$.

This completes the proof of the fact that $(DQ)(f)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$ if and only if $\kappa_j \ge -1$ $(1 \le j \le N)$. □

5. PROOF OF THE SECOND PART OF THEOREM 3.1

By the properties of Θ the map L given by $(Lw)(\zeta) = [i\Theta(\zeta)^{-1}]w(\zeta)$ $(\zeta \in b\Delta)$ is a Banach space isomorphism of $(\mathcal{A}^\alpha)^N$ onto itself and

$$L(\mathrm{Ker}(DQ)(f)) = \{w \in (\mathcal{A}^\alpha)^N : \Re[M(\zeta)w(\zeta)] = 0 \ (\zeta \in b\Delta)\}.$$

Now, $\Re[M(\zeta)w(\zeta)] = 0$ $(\zeta \in b\Delta)$ is equivalent to $w(\zeta) = M(\zeta)^{-1}\overline{M(\zeta)w(\zeta)}$ $(\zeta \in b\Delta)$, that is, to $w(\zeta) = \Lambda(\zeta)\overline{w(\zeta)}$ $(\zeta \in b\Delta)$. If $\kappa_j \ge 0$ then the general solution $w_j \in \mathcal{A}^\alpha$ of

$$w_j(\zeta) = \zeta^{\kappa_j}\overline{w_j(\zeta)} \quad (\zeta \in b\Delta) \tag{5.1}$$

is

$$w_j(\zeta) = (t_1^j + it_2^j) + (t_3^j + it_4^j)\zeta + \cdots + (t_3^j - it_4^j)\zeta^{\kappa_j - 1} + (t_1^j - it_2^j)\zeta^{\kappa_k}$$

where t_k^j are arbitrary real constants and so the space of all solutions $w_j \in \mathcal{A}^\alpha$ of (5.1) has dimension $\kappa_j + 1$. If $\kappa_j < 0$ then the only solution $w_j \in \mathcal{A}^\alpha$ of (5.1) is $w_j = 0$. This shows that $\mathrm{Ker}(DQ)(f)$ is always finite dimensional and if $\kappa_j \ge -1$ $(1 \le j \le N)$ then its dimension is $\kappa + N$.

Suppose that $\kappa_j \ge -1$ $(1 \le j \le N)$. Let a neighbourhood $U \subset (\mathcal{A}^\alpha)^N$ of f be so small that $\varphi(b\Delta) \subset V$ for every $\varphi \in U$. Define the map $P : [\mathcal{C}^2(V)]^N \times U \to (\mathcal{C}^\alpha)^N$ by $P(r, \varphi)(\zeta) = r(\varphi(\zeta))$ $(\zeta \in b\Delta)$. By a generalization of a result of Hill and Taiani [Gl, Lemma 5.1] P is of class $\mathcal{C}^1$. The partial derivative $(D_Y P)(\rho, f)$ with respect to $Y = (\mathcal{A}^\alpha)^N$ equals $(DQ)(f)$. By Section 4, $(DQ)(f)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$ and by the preceding discussion, $\mathrm{Ker}(DQ)(f)$ is of (finite) dimension $\kappa + N$ and thus complemented in $(\mathcal{A}^\alpha)^N$. The second part of Theorem 3.1 now follows from the implicit mapping theorem as stated in Section 2.

This completes the proof of Theorem 3.1. $\square$

6. GENERALIZATIONS AND REMARKS ABOUT THE NONORIENTABLE CASE

Let M and f be as above. Choose a small $\tau > 0$. For each $\zeta \in b\Delta$ take $\{x \in M : |x - f(\zeta)| < \tau\}$, the τ-neighbourhood of $f(\zeta)$ in M and translate it to the origin to get

$$M(\zeta) = -f(\zeta) + \{x \in M : |x - f(\zeta)| < \tau\}.$$

If g is sufficiently close to f then $g(b\Delta) \subset M$ if and only if

$$h(\zeta) \in M(\zeta) \quad (\zeta \in b\Delta) \tag{6.1}$$

where $h = g - f$. Thus, to find all analytic discs $g \in (\mathcal{A}^\alpha)^N$ close to f which satisfy $g(b\Delta) \subset M$ is the same as to find all maps $h \in (\mathcal{A}^\alpha)^N$ close to 0 such that (6.1) holds.

Suppose now that for each $\zeta \in b\Delta$, $M(\zeta)$ is a maximal real submanifold of $B = \{x \in \mathbb{C}^N : |x| < \tau\}$ which passes through the origin and suppose that we want to find all maps $h \in (\mathcal{A}^\alpha)^N$ close to 0 such that (6.1) holds. In the case when for each $\zeta \in b\Delta$, $M(\zeta)$ is a real-linear subspace of $\mathbb{C}^N$ and when the bundle $\{M(\zeta) : \zeta \in b\Delta\}$ is trivial (6.1) reduces to the classical Riemann–Hilbert problem [Ve1]. We shall assume that for each $\zeta \in b\Delta$,

$$M(\zeta) = \{x \in B : \rho(\zeta, x) = 0\} \tag{6.2}$$

where $\rho(\zeta, \bullet) \in [\mathcal{C}^2(B)]^N$, $\partial\rho_1 \wedge \cdots \wedge \partial\rho_N \neq 0$ on B, $\rho(\zeta, 0) = 0$ $(\zeta \in b\Delta)$ and where the functions $\rho(\zeta, \bullet)$ (and thus the manifolds $M(\zeta)$) depend α-smoothly on ζ, that is, the map $\zeta \mapsto \rho(\zeta, \bullet)$ belongs to $\mathcal{C}^\alpha(b\Delta, \mathcal{C}^2(B))^N$ [Gl].

If $U \subset (\mathcal{A}^\alpha)^N$ is a sufficiently small neighbourhood of 0 then the map $Q : h \mapsto (\zeta \mapsto \rho(\zeta, h(\zeta)))$ is a $\mathcal{C}^1$ map from U to $(\mathcal{C}^\alpha)^N$ [Gl] and $Q(h) = 0$ if and only if $h(\zeta) \in M(\zeta)$ $(\zeta \in b\Delta)$. Only minor modifications of the proof of Theorem 3.1 are now necessary to prove.

Theorem 6.1. *Let ρ and Q be as above. For each $\zeta \in b\Delta$ let $T(\zeta)$ be the tangent space of $M(\zeta)$ at 0. Let $\kappa_1, \ldots, \kappa_N$ be the partial indices of the (trivial) bundle $\{T(\zeta) : \zeta \in b\Delta\}$ and let κ be its total index. Then $(DQ)(0)$ maps $(\mathcal{A}^\alpha)^N$ onto $(\mathcal{C}^\alpha)^N$ if and only if $\kappa_j \geq -1$ $(1 \leq j \leq N)$. If this happens then there are a neighbourhood $\mathcal{P} \subset \mathcal{C}^\alpha(b\Delta, \mathcal{C}^2(B))^N$ of ρ and a neighbourhood $W \subset (\mathcal{A}^\alpha)^N$ of 0 such that*

(i) *$\{(r, g) \in \mathcal{P} \times W : r(g(\zeta)) = 0$ $(\zeta \in b\Delta)\}$ is a $\mathcal{C}^1$ submanifold of $\mathcal{P} \times W$;*

(ii) *for each $r \in \mathcal{P}$, $\{g \in W : r(g(\zeta)) = 0$ $(\zeta \in b\Delta)\}$ is a $\mathcal{C}^1$ submanifold of W of dimension $\kappa + N$.*

Corollary 6.2. *If the partial indices κ_j of the (trivial) bundle T satisfy $\kappa_j \geq -1$ $(1 \leq j \leq N)$ then the family of analytic discs $g \in U$ such that $g(\zeta) \in M(\zeta)$*

$(\zeta \in b\Delta)$ *is a* C^1 *submanifold of* W *of dimension* $\kappa + N$ *where* κ *is the total index of* T.

Suppose now that the bundle T is nontrivial, so that there is no global function ρ satisfying (6.2) and the functions $\zeta \mapsto \rho(\zeta, \bullet)$ defining $M(\zeta)$ are only given locally. Assume that these functions are of class C^1. We now indicate how to deduce the appropriate analogue of Corollary 6.2 from Corollary 6.2. Let $\kappa_1, \ldots, \kappa_N$ be the partial indices of T and let κ be its total index. Clearly κ is odd as T is nontrivial. Define

$$M_1(e^{i\theta}) = \{te^{-i\theta/2} : t \in R\} \quad (\theta \in R).$$

Now, $\{M(\zeta) : \zeta \in b\Delta\}$ is a nontrivial bundle of real lines in $\mathbb{C}$ whose index is -1. For each $\zeta \in b\Delta$ define $\tilde{M}(\zeta) \subset \mathbb{C}^N \times \mathbb{C}$ by

$$\tilde{M}(\zeta) = M(\zeta) \times M_1(\zeta) \quad (\zeta \in b\Delta).$$

For each ζ, $\tilde{M}(\zeta)$ is a maximal real submanifold of $B \times \mathbb{C}$. Denote by $\tilde{T}(\zeta) = T(\zeta) \times M_1(\zeta)$ the tangent space of $\tilde{M}(\zeta)$ at 0. The bundle $\tilde{T}$ is trivial and its partial indices are $\kappa_1, \ldots, \kappa_N, -1$. Since $\tilde{T}(\zeta)$ are totally real subspaces of $\mathbb{C}^{N+1}$ of dimension $N+1$ the bundle $\{T(\zeta)^{\perp} : \zeta \in b\Delta\}$ is also trivial (here $\perp$ denotes the (real) orthogonal complement in $\mathbb{C}^N$). It is now easy to see that the required smoothness of defining functions implies the existence of a global defining function $\tilde{\rho}$, that is, there are a ball $\tilde{B} \subset \mathbb{C}^{N+1}$ centered at 0 and a map $\tilde{\rho} \in C^1(b\Delta, C^2(\tilde{B}))^{N+1}$ such that $\tilde{M}(\zeta) \cap \tilde{B} = \{x \in \tilde{B} : \tilde{\rho}(\zeta, x) = 0\}$ for each $\zeta \in b\Delta$.

If $h \in \mathcal{A}^\alpha$ satisfies $h(\zeta) \in M_1(\zeta)$ $(\zeta \in b\Delta)$ then $h(\zeta) = \overline{\zeta h(\zeta)}$ $(\zeta \in b\Delta)$ so $h = 0$. This implies that $\tilde{g} \in (\mathcal{A}^\alpha)^{N+1}$ satisfies $\tilde{g}(\zeta) \in \tilde{M}(\zeta)$ $(\zeta \in b\Delta)$ if and only if $\tilde{g} = (g, 0)$ where $g \in (\mathcal{A}^\alpha)^N$ satisfies $g(\zeta) \in M(\zeta)$ $(\zeta \in b\Delta)$. Indeed, if $\tilde{g} = (g, h)$ then $\tilde{g}(\zeta) \in M(\zeta)$ $(\zeta \in b\Delta)$ if and only if $g(\zeta) \in M(\zeta)$, $h(\zeta) \in M_1(\zeta)$ $(\zeta \in b\Delta)$, that is, if and only if $g(\zeta) \in M(\zeta)$ $(\zeta \in b\Delta)$ and $h = 0$.

By Corollary 6.2 there are neighbourhoods U of 0 in $(\mathcal{A}^\alpha)^N$, U_1 of 0 in $\mathcal{A}^\alpha$ such that the set of all $\tilde{g} \in U \times U_1$ such that $\tilde{g} \in \tilde{M}$ $(\zeta \in b\Delta)$ is a C^1 submanifold of $U \times U_1$ of dimension $(\kappa - 1) + (N + 1) = \kappa + N$. Since all such $\tilde{g}$ are contained in $(\mathcal{A}^\alpha)^N = (\mathcal{A}^\alpha)^N \times \{0\}$ it follows that the family of all discs $g \in U$ such that $g(\zeta) \in M(\zeta)$ $(\zeta \in b\Delta)$ is a C^1 submanifold of U of dimension $\kappa + N$.

This implies the following corollary in the nonorientable case.

Corollary 6.3. *Let* M *be a maximal real submanifold of* $\mathbb{C}^N$ *of class* C^2 *and let* $f : \Delta \to \mathbb{C}^N$ *be an analytic disc such that* $f(b\Delta) \subset M$. *For each* $\zeta \in b\Delta$ *let* $T(\zeta)$ *be the tangent space of* M *at* $f(\zeta)$ *and let* $\kappa_1, \ldots, \kappa_N$ *be the partial indices of the bundle* $\{T(\zeta) : (\zeta \in b\Delta)\}$. *If* $\kappa_j \geq -1$ $(1 \leq j \leq N)$ *then there is a neighbourhood* $U \subset (\mathcal{A}^\alpha)^N$ *of* f *such that the family of analytic discs* $g \in U$ *such that* $g(b\Delta) \subset M$ *is a* C^1 *submanifold of* U *of dimension* $N + \kappa$ *where* κ *is the total index of* T.

In the orientable case this, together with a stability result, follows from

Theorem 3.1. In the nonorientable case one obviously cannot work with global defining functions as we did in Theorem 3.1 and a more sophisticated approach is necessary to prove the smooth dependence of the family of analytic discs on the manifold. Note, however, that (3.1) is stable with respect to small perturbations of M and f [Ve2] which implies that the conclusion of Corollary 6.3 still holds if we replace M and f with sufficiently small perturbations.

ACKNOWLEDGEMENT

The author is indebted to J.-M. Trepreau for a very stimulating discussion and for sending him a paper in preparation [T]. He is also indebted to S. Baouendi and L. Rothschild for sending the preprint [BRT], to Y.-G. Oh for sending the preprints [O1], [O2], and to M. Černe, S. Ivashkovich and N. Shcherbina for stimulating discussions.

This work was supported in part by a grant from the Ministry of Science and Technology of the Republic of Slovenia.

Added in proof. S. Baouendi told the author that Andrei Vityaev noticed that the part of Theorem 3.1 proved in Section 4 also follows from the theorem about solvability of non-homogeneous Hilbert problem [Ve2, p. 51, Theorem 4].

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